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PROPAGATION OF A PLANE JET OF CONDUCTING FLUID

K. E. Dzhaugashtin

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An exact solution is given for the propagation of a plane jet of conducting liquid in a magnetic field at small values of magnetic Reynolds number. Profiles of velocity, temperature and intensity of magnetic field are given.

We shall investigate the laminar motion of an incompressible conducting fluid issuing from a plane jet source, assuming that the physical properties of the fluid in the jet and in the surrounding medium are the same and that the magnetic Reynold numbers are much smaller than one. We shall also assume that the lines of force of the external magnetic field in the flow plane xy are normal to the direction of fluid motion (Fig. 1).

The initial system of boundary layer equations for this case has been given in [1]. We shall write this system in the following form:

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}-\frac{\sigma\mu H_y^2}{\rho}u,\quad \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{1}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=a\frac{\partial^2 T}{\partial y^2}+\frac{\sigma\mu H_y^2}{\rho}u^2, \tag{2}$$

$$\frac{dH_x}{dy}=-\sigma H_y u. \tag{3}$$

The first two equations (1) are the equations of fluid motion. The velocity profiles obtained by solving these equations will be used for the subsequent determination of the temperature and magnetic intensity fields in the jet on the basis of the equations of heat propagation (2) and induction (3), respectively.

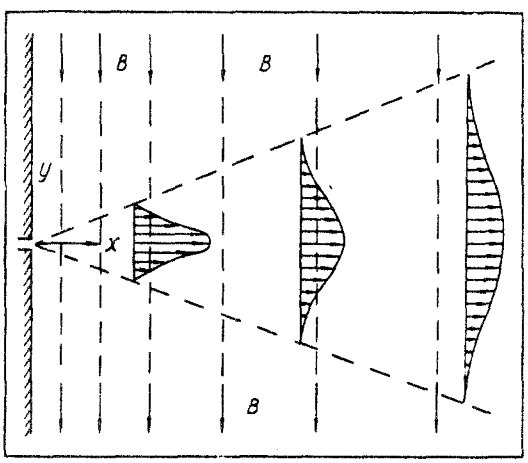

Fig. 1. Laminar plane free jet in a magnetic field.

The dynamic part of this problem has been investigated by several authors [2-4]. An error in [2] was pointed out in [3] and [4]. An expression for a universal velocity distribution function was obtained in [3]. Lack of an "integral condition," however, prevented the author from arriving at final formulas for the velocity components.

A solution of the dynamic and thermal problems is given below, and the necessary "integral condition" is obtained. The intensity profile of the induced component of the magnetic field is also found in the approximation $Re_m \ll 1$.

We shall confine ourselves to finding a self-similar solution to the above problem. We write the exponential transformation equations in the form

$$u=u_m F'(\varphi),\ u_m=Ax^{\alpha},\ \varphi=Bx^{\beta}y,\ H_y=H_0x^{p}. \tag{4}$$

The solution of the equation of motion must satisfy the boundary conditions

$$\frac{\partial u}{\partial y}=0,\ v=0\quad \text{at}\quad y=0,$$

$$u=0,\ \frac{\partial u}{\partial y}=0\quad \text{at}\quad y=\pm\infty.$$

To integrate the equation we substitute the expression for the velocity components and their derivatives from (4) into the first equation of system (1). We then obtain the following ordinary differential equation for the relative velocity profile:

$$F'''+3(\alpha+1)FF''-6\alpha F'^2-NF'=0 \tag{5}$$

with boundary conditions

$$F=0,\ F'=1,\ F''=0 \quad \text{at} \quad \varphi=0,$$
$$F'=0,\ F''=0 \quad \text{at} \quad \varphi=\pm\infty. \tag{6}$$

In deriving (5) it was assumed that

$$\beta=p=\frac{\alpha-1}{2},\quad \frac{A}{\nu B^2}=6,\ N=\frac{6\sigma\mu H_0^2}{A\rho}. \tag{7}$$

The first of these relations was found from the condition that the dimensionless velocity profile $F'(\varphi)$ be independent of φ, the second from the fact that the constants A and B are arbitrary, while in the third relation N denotes the dimensionless complex characterizing the magnetic effect on the flow.

To determine the constant α, we integrate (5) between 0 and ∞ to obtain

$$\alpha=-\frac{1}{3}\left(1+\frac{N}{3}\,\frac{F\,\infty}{\int\limits_0^\infty [F'(\varphi)]^2\,d\varphi}\right). \tag{8}$$

After substituting (8) into (5), we rewrite the latter in the form

$$F'''+2(F''+F'^2)-\frac{N}{3}F(\infty)\left(\int\limits_0^\infty [F'(\varphi)]^2\,d\varphi\right)^{-1}\left(FF''-2F'^2+3\frac{1}{F(\infty)}\int\limits_0^\infty [F'(\varphi)]^2\,d\varphi\,F'\right)=0. \tag{9}$$

When N = 0, Eq. (9) goes over into the equation for an ordinary hydrodynamic jet, whose solution is the function

$$F'=1-\operatorname{th}^2\varphi. \tag{10}$$

It is easy to verify that function (10) satisfies Eq. (9), even for nonzero values of N.

Using (10), we obtain from (8) a final expression for the exponent

$$\alpha=-\frac{1}{3}\left(1+\frac{N}{2}\right). \tag{11}$$

The expression obtained for the relative velocity profile must be related to the actual velocity field. In jet-source theory some "conservation condition" is ordinarily used for this purpose. To obtain such a condition in the case of flow of a conducting fluid, it is convenient to employ the integral $\int\limits_{-\infty}^{\infty} u^{\beta/\alpha}\,dy=\text{const}$, which does not depend on the longitudinal coordinate for transformations (4). This invariant was proposed in [5] for solving the hydrodynamic problem. To obtain a similar integral condition from the differential equations, we multiply the first equation of (1) by $u^{\delta-2}$ and add this to the continuity equation after first multiplying by $u^{\delta-1}$.

Integrating the sum over the cross section of the jet, we obtain

$$\frac{\partial}{\partial x}\left[\frac{1}{\delta-1}\int\limits_{-\infty}^{\infty} u^{\delta}\,dy\right]=\int\limits_{-\infty}^{\infty}\left(\nu u^{\delta-2}\frac{\partial^2 u}{\partial y^2}\,dy-\frac{\sigma\mu H_y^2}{\rho}\,u^{\delta-1}\right)dy,\quad \delta=\frac{\beta}{\alpha}. \tag{12}$$

Evaluating the integral on the right using (4), (11), and (5), we see that it is identically equal to zero. In this case we obtain from (12) the following integral conservation condition:

$$\int\limits_{-\infty}^{\infty} u^{\delta}\,dy=(\delta-1)\,D. \tag{13}$$

The constant of integration D depends, generally speaking, on the momentum J of the jet, the density ρ, the conductivity σ of the fluid, and the magnetic field intensity H_y. In the limiting case of a nonmagnetic jet ($\sigma_0 = 0$ or $H_0 = 0$) the constant $D = J_0/\rho$, i.e., coincides with the kinematic momentum of the jet ($\delta = 2$). The value of D should be assumed given, as is usual in jet-source problems. It should be noted that the integral relation derived determines the region of variation of the self-similarity constants, since δ can take values in the range $1 \le \delta \le 2$, as may be seen from (13).

We now determine the constants A and B by substituting (4) into (13) and using (7) and (11). After some transformations we obtain

$$A^{\delta}/B = \lambda(\delta)(\delta - 1)D, \quad \lambda(\delta) = \left\{ \int_{-\infty}^{\infty} [F'(\varphi)]^{\delta}\, d\varphi \right\}^{-1}. \tag{14}$$

From the relation $A = 6\nu B^2$ and (14), we find the required expressions:

$$A = \left[\frac{(\delta - 1)\lambda(\delta) D}{\sqrt{6\nu}}\right]^{\frac{2}{2\delta-1}}, \quad B = \frac{1}{\sqrt{6\nu}} \left[\frac{(\delta - 1)\lambda(\delta) D}{\sqrt{6\nu}}\right]^{\frac{1}{2\delta-1}}. \tag{15}$$

Finally, we write the relations for the variation of the mass flow G per second and momentum J along the jet:

$$G = \int_{-\infty}^{\infty} \rho u\, dy = 12\rho\nu B x^{\frac{1}{3}(1-N/4)}, \tag{16}$$

$$J = \int_{-\infty}^{\infty} \rho u^2 dy = 48\rho\nu^2 B^3 x^{N/4}. \tag{17}$$

Let us briefly consider the results obtained. When a jet of conducting fluid propagates in a magnetic field, the field impedes the motion. As the electromagnetic volume force increases, the velocity drop (11) along the jet axis occurs more rapidly, and the mixing region increases. In this case, however, the relative velocity profile (10) does not change relative to the case N = 0.

The constants denoting the variation along the jet axis of the velocity α and the effective jet width β, like the value of the magnetic interaction parameter N and the ratio $\delta = \beta/\alpha$, are determined by the value of the constant p (from the expression $H_y = H_0 x^p$):

$$\alpha = 1 + 2p, \quad \beta = p, \quad \delta = p/(1+2p), \quad N = -4(2+3p). \tag{18}$$

From expression (15) for A it follows that the value of the constant lies in the range $1 \leq \delta \leq 2$. Since all the constants are interrelated, specific values of the self-similarity constants α, β, etc., correspond to each value of δ in this range.

For clarity, these constants have been tabulated for several values of the parameter N. The values in the top row of the table correspond to ordinary hydrodynamic flow (N = 0, $\delta = 2$; values of $\delta > 2$ lead to the unreal case of negative values of N). At values of the constants corresponding to the bottom row of the table, Eq. (1) has a trivial solution equal to zero (15) (when N = 4, $\delta = 1$; the case $\delta < 1$ does not give flow, since then A < 0). Note that the last case of a "degenerate jet" is naturally related to the self-similar solution examined here.

Values of Self-Similarity Constants

N	α	$\beta = p$	δ	γ
0	—1/3	—2/3	2	—4/3
1	—1/2	—3/4	3/2	—1/4
2	—2/3	—5/6	5/4	—1/6
3	—5/6	—11/12	11/10	—1/12
4	—1	—1	1	0

As far as the integral characteristics of the jet are concerned, as the magnetic field increases, the ejecting properties of the jet decrease, and mass addition along the length of the jet proceeds more slowly (16). The momentum of the jet also decreases as the magnetic field H_0 increases. In this case, since at the origin the magnetic field is infinitely large (2), the jet must have infinite momentum (17), in contrast to the case of a nonconducting fluid. As a result of interaction with the magnetic field, at an infinitely small distance the momentum becomes finite. These characteristics at the origin ($H_y \to \infty$ and $I \to \infty$) are a generalization of the characteristics of a nonmagnetic jet source.

Let us now turn our attention to the following important fact. It follows from (18) that the value of the magnetic interaction parameter N depends on the value of the constant p. Expanding the expression for N (7), using (15) and (18), we put it in the form

$$\frac{\sigma\mu H_0^2}{\rho} = -\frac{4}{6}(2+3p)\left(-\frac{1+p}{1+2p}\,\frac{\lambda D}{\sqrt{6\nu}}\right)^{-(1+2p)}. \tag{19}$$

Equation (19) relates the external parameters of the problem, the momentum of the jet J, the field intensity H_y, and p, and determines the condition for self-similar flow in the jet. It follows, in particular, that, for a given jet momentum, for example, the external field cannot be chosen arbitrarily, and vice versa. This is a characteristic of the motion of jets of conducting fluid in a magnetic field at small values of the magnetic Re number [6]. Pure hydrodynamic jet flow at a great distance from the source is known to be always self-similar.

Let us now derive an expression for the induced longitudinal component of the magnetic field H_x. From (3), using the expression for the velocity profile (10) and taking account of the symmetry of the lines of force about the jet axis (H_x = 0 at y = 0), we find

$$h(\varphi) = H_x/H_{x\infty} = \operatorname{th}\varphi, \quad H_{x\infty} = -\sigma H_0 B x^{-\frac{N}{4}}.$$

The component H_x is antisymmetric relative to the jet axis. At its boundaries the H_x are opposite in direction but equal in magnitude, which is determined by the total current I_0 flowing in the cross section of the jet:

$$I_0 = \int_{-\infty}^{\infty} j dy, \quad (j = \sigma u B).$$

Note that the total current is proportional to the mass flow per second of the jet:

$$I_0 = \frac{\sigma \mu H_0}{\rho} G.$$

We shall examine the heat propagation equation for two variants of the boundary condition:

$$\frac{\partial T}{\partial y} = 0 \quad \text{at} \quad y = 0, \; T = T_\infty \quad \text{at} \quad y = \pm\infty;$$

$$T = T_1 \quad \text{at} \quad y = +\infty, \quad T = T_2 \quad \text{at} \quad y = -\infty.$$

We introduce the self-similarity transformation:

for a symmetric thermal boundary layer

$$T - T_\infty = (T_m - T_\infty)\Theta_1(\varphi), \; T_m - T_\infty = \Gamma x^\gamma; \tag{20}$$

for an "asymmetric" thermal boundary layer

$$T - T_2 = (T_1 - T_2)\Theta_2(\varphi) \quad (\text{i.e., } \gamma = 0). \tag{21}$$

For a solution remote from the source, we may neglect the term in (3) relating to the Joule dissipation, which decreases with distance from the source faster than the remaining terms.

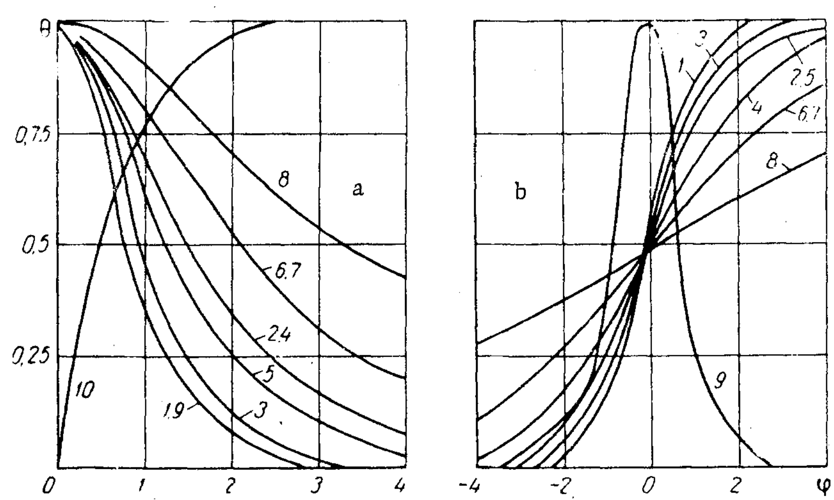

Fig. 2. Relative velocity, magnetic field and temperature profiles in in a plane jet (a $-\Theta_1(\varphi)$, b $-\Theta_2(\varphi)$): 1) N = 0 and Pr = 1; 2) 0 and 0.5; 3) 1 and 1; 4) 1 and 0.5; 5) 2 and 1; 6) 2 and 0.5; 7) 3 and 1; 8) N = 4; 9) $F'(\varphi)$; 10) $h(\varphi) = -h(-\varphi)$.

From (3), taking into account (4) and (20), we have

$$\Theta'' + 3\Pr[(\alpha+1)F\Theta' - 2\gamma F'\Theta] = 0. \tag{22}$$

Integrating (22) within infinite limits, we find $\gamma = -(\alpha+1)/2$ for boundary conditions

$$\Theta' = 0 \quad \text{at} \quad \varphi = 0,$$

$$\Theta = \Theta' = 0 \quad \text{at} \quad \varphi = \pm\infty. \tag{23}$$

Substituting the value of γ in (22), we rewrite it in the form

$$\Theta'' + 3\mathrm{Pr}(\alpha + 1)\ (F\Theta)' = 0. \tag{24}$$

Integration of (24) with boundary conditions (23) gives

$$\Theta_1(\varphi) = [F'(\varphi)]^{\frac{3}{2}\mathrm{Pr}(1+\alpha)} = [\mathrm{ch}\ \varphi]^{-\mathrm{Pr}(2-N/2)}\ . \tag{25}$$

Note that when Pr = 1 - N/4, the temperature and velocity profiles are similar.

Using the integral conditions of conservation of excess heat content

$$Q = \int_{-\infty}^{\infty} \rho C_p u (T - T_\infty)\, dy,$$

we determine the constant Γ:

$$\Gamma = \frac{B}{A} \frac{Q}{\rho C_p} \left[\int_{-\infty}^{\infty} [\mathrm{ch}\varphi]^{-\mathrm{Pr}\left(2-\frac{N}{2}\right)-2} d\varphi \right]^{-1}.$$

In the second case (γ = 0)

$$\Theta = 1 \quad \text{at} \quad \varphi = +\infty,$$
$$\Theta = 0 \quad \text{at} \quad \varphi = -\infty.$$

From the equation

$$\Theta'' + 3\mathrm{Pr}(\alpha + 1) F' \Theta = 0 \tag{26}$$

we have

$$\Theta_2(\varphi) = \left[\int_{-\infty}^{\infty} [\mathrm{ch}\ \varphi]^{-\mathrm{Pr}(2-N/2)} d\varphi \right]^{-1} \left[\int_{-\infty}^{\varphi} [\mathrm{ch}\ \varphi]^{-\mathrm{Pr}(2-N/2)} d\varphi \right]. \tag{27}$$

In both cases the magnetic field acts via variation of the velocity field, and the width of the thermal boundary layer increases with increase in the magnetic interaction parameter N (Fig. 2).

Naturally, when there is no magnetic field (N = 0), all the solutions obtained above go over into the usual hydrodynamic solutions.

NOTATION

u, v – longitudinal and transverse components of velocity; x, y – longitudinal and transverse coordinates; T – temperature; H – magnetic field intensity; $\alpha, \beta, \gamma, \delta, A, B, \Gamma$ – constants to be determined; D – a constant; G – mass flow in jet; J – momentum of jet; Q – excess heat content of jet; I_0 – total current; H_0 – reference value of magnetic field intensity; p – a constant; σ – conductivity; ν – kinematic viscosity; ρ – density; μ – magnetic permeability; $F'(\varphi)$ – velocity; φ – coordinate; Re_m – magnetic Reynolds number; N – magnetic interaction parameter; θ_1, θ_2 – temperature for symmetric and asymmetric thermal boundary conditions.

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